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SHORT NOTES

A simple construction for shear stress

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Abstract—On a plane of given orientation, the direction of resolved shear stress (τ) is controlled by the orientation of the principal stress axes and by the ratio of principal stress differences. A simple stereographic construction is presented for the determination of the direction of τ.

INTRODUCTION

IT IS frequently necessary to calculate the direction of resolved shear stress acting on a plane obliquely inclined to the principal stress axes, especially in work relating to petrofabrics, rock mechanics and fault slip analysis. A simple stereographic construction is presented below based on a geometrical property of the stress quadric. It is less involved than the Mohr circle construction of Zizicas (1955) and the geometrical construction given by Johnson & Mellor (1973). Unlike the methods suggested by Goodman (1963) and Jaeger & Cook (1979) the proposed construction involves the minimum of calculation and yields a solution directly in stereographic projection.

A USEFUL PROPERTY OF THE ELLIPSOID

At any point on the surface of an ellipsoid, the plane containing the radius R *and normal to the surface is perpendicular to the plane on which* R *is a principal axis of the sectional ellipse.*

To verify this theorem we refer to the concept of conjugate radii of an ellipsoid and their well-known properties. Conjugate radii are parallel to three diameters of an ellipsoid, such that the plane through any two of them (a diametral plane) bisects all chords parallel to the third (Macaulay 1930, pp. 85–89). The following two rules apply to conjugate radii. Firstly, a pair of conjugate radii are only perpendicular when they coincide with the principal axes of the elliptical section on the diametral plane defined by those two radii. Secondly, the tangent planes of the ellipsoid at the extremities of a diameter are parallel to the diametral plane which is conjugate to that diameter.

Figure 1 shows a radius R and tangent plane at point P on an ellipsoid's surface. Plane RN, the plane containing R and the normal to the tangent plane N, has O as its normal. As stated above the tangent plane is conjugate to R. As a result, any line parallel to the tangent plane potentially belongs to a set of conjugate radii to which R belongs. Such a line parallel to the tangent plane is the line O and hence the plane containing O and R can be considered to be a diametral plane containing two out of a set of three conjugate radii for the ellipsoid. We note that O and R are mutually perpendicular and hence that O and R must be the principal axes of the ellipse on the section plane of the ellipsoid which is parallel to O and R. This plane containing O and R is perpendicular to the plane containing R and N and therefore the statement above is shown to be valid.

THE STRESS QUADRIC OF CAUCHY

For a deviatoric state of stress, the magnitude of the normal stress component is influenced by the orientation

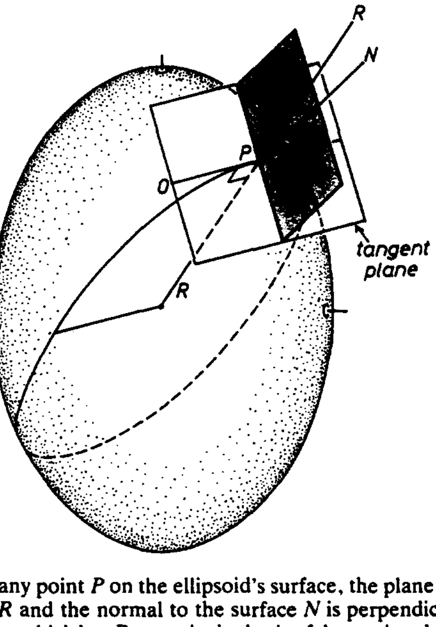

Fig. 1. At any point P on the ellipsoid's surface, the plane containing the radius R and the normal to the surface N is perpendicular to the section plane which has R as a principal axis of the sectional ellipse. See text for proof of this statement.

of the plane on which the stress acts. This is conveniently illustrated by the geometry of the quadric surface representing the equation:

$$\sigma_1 x^2 + \sigma_2 y^2 + \sigma_3 z^2 = \pm 1$$

(Durelli *et al.* 1958, p. 68); when $\sigma_1 > \sigma_2 > \sigma_3 > 0$, the surface is an ellipsoid. We are discussing shear stress which is not affected by the absolute magnitude of the principal stresses, only by principal stress differences. Consequently, for present purposes, stress states involving negative principal stresses can also be represented by an ellipsoid after the addition of an appropriate hydrostatic component (a constant added to each principal stress value). For any plane with a normal (n) of given orientation, the radius R of the ellipsoid parallel to that normal (n) is inversely proportional to the square root of the value of the normal stress (σ) acting on the plane (Fig. 2). The normal to the ellipsoid, N, drawn from the point where R intersects its surface, points in the direction of the resultant stress (s) acting on the plane. This ellipsoid should not be confused with the stress ellipsoid of Lamé which has distinct properties (Durelli *et al.* 1958, p. 65).

To find the direction of shear stress (τ) acting on a plane we can now make use of the ellipsoid property demonstrated in the previous section. Namely, it has been shown that the plane containing $R//n$ and $N//s$ is perpendicular to the section plane having $R//n$ as the major or minor axis of the ellipse (Fig. 3). We also know that the shear stress direction labelled τ is given by the intersection of the plane ($\sigma\tau$) on which the resultant stress acts (Fig. 3). τ is also perpendicular to the elliptical section which has $R//n$ as a principal axis. This section plane can be found stereographically by a modification of the Biot–Fresnel construction (Bloss 1961, p. 161).

FINDING THE ELLIPTICAL SECTION PLANE WHICH HAS $R//n$ AS A PRINCIPAL AXIS

We start by finding the planes of circular section through the Cauchy ellipsoid. The two circular sections

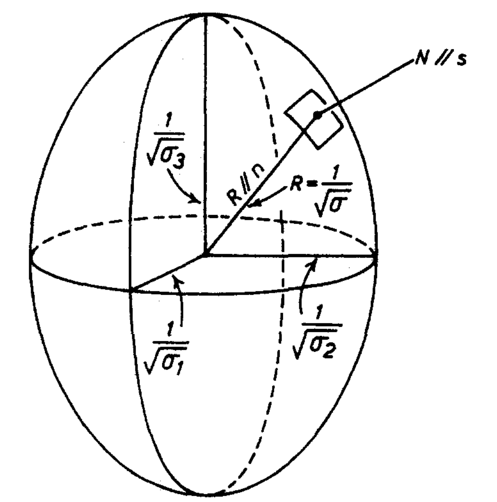

Fig. 2. The stress quadric of Cauchy.

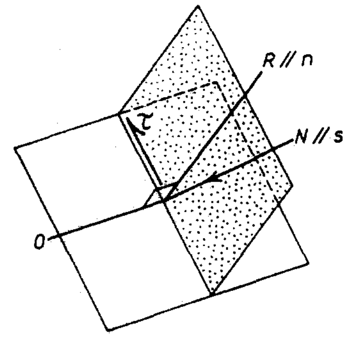

Fig. 3. The shear stress direction τ in the plane Oτ is given by the intersection of this plane with the plane RN. R the radius of the Cauchy ellipsoid, is also parallel to the normal n of the plane Oτ. N, the normal to the ellipsoid, is parallel to the direction of resultant stress (s) acting on plane Oτ.

of an ellipsoid with principal radii $a > b > c$ intersect along the b axis and are inclined to the c axis at an angle V, where

$$\cot^2 V = \frac{\dfrac{1}{b^2} - \dfrac{1}{a^2}}{\dfrac{1}{c^2} - \dfrac{1}{b^2}}$$

(see for example, Flinn 1962). The Cauchy ellipsoid has its a axis parallel to σ_3 ($a = 1/\sqrt{\sigma_3}$), its b axis parallel to σ_2 ($b = 1\sqrt{\sigma_2}$) and c axis parallel to σ_1 ($c = 1/\sqrt{\sigma_1}$) and hence has circular sections which are inclined to the σ_1 axis at an angle of V, where

$$\cot^2 V = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_2}. \qquad (1)$$

The quantity $(\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_2)$ has been referred to as the stress ratio, **R** (Lisle 1980) and, it is implicit in Bott's (1959) equation that this ratio controls the directions of shear stress on a given plane.

The stress diagram in Fig. 4, shows lines representing the equation $V = \cot^{-2}$ (stress ratio, **R**). This diagram can be used to find the V angle for any given stress state.

The Biot–Fresnel construction (Bloss 1961, p. 161) allows the principal directions of an elliptical section to

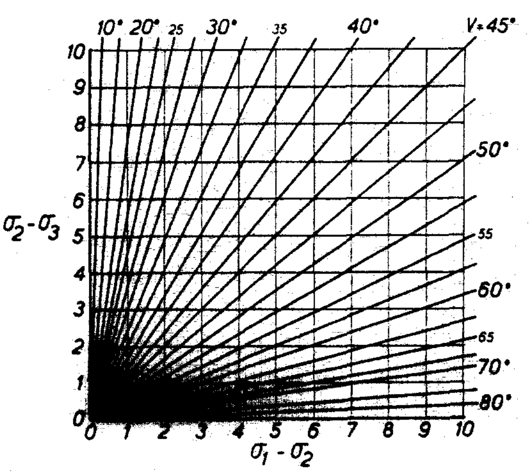

Fig. 4. The stress diagram (Lisle 1980) allows the angle V for the Cauchy ellipsoid to be found for any stress state. Stresses in any units may be plotted providing same scaling is used on both axes of the diagram.

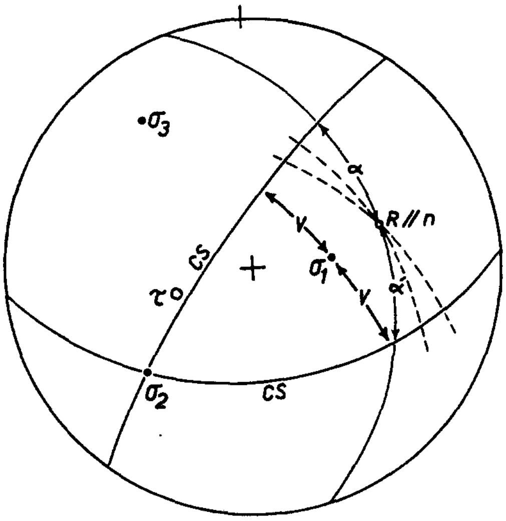

Fig. 5. The stereographic construction of the shear stress direction τ on a plane with normal $R//n$. The planes CS are the two circular sections of the Cauchy ellipsoid. See text for explanation.

be found. These are given by the acute and obtuse bisectors of the angle in the section plane made by the two lines which are the intersections with each of the circular sections. To find the elliptical section which has $R//n$ as a principal axis, we consider on the stereogram (Fig. 5) a variety of planes passing through $R//n$, until the latter bisects the angle between the intersections with circular sections. In terms of Fig. 5, we find the plane for which $\alpha = \alpha'$.

When found, this plane corresponds to the plane containing R and O in Fig. 3 and is the plane which is perpendicular to the shear stress direction τ in the plane being considered (Fig. 5).

SUMMARY

To find the direction of shear stress on a given plane, the following steps are followed.

(1) Plot σ_1, σ_2, σ_3 axes together with $R//n$, the normal to the given plane, on a stereogram (Fig. 5).

(2) Using Fig. 4 or equation (1), determine the angle V for the Cauchy ellipsoid and plot the two circular sections on the stereogram.They intersect along σ_2 and are inclined at an angle V from σ_1.

(3) Find the plane passing through $R//n$ and which has $R//n$ as the bisector of the traces of the circular sections (Fig. 5).

(4) The direction perpendicular to the plane found in (3) is the direction of shear stress (τ) in the plane which has the normal $R//n$.

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